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PRODUCT-PRESERVING FUNCTORS ON SMOOTH MANIFOLDS

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Functors from the category of connected smooth manifolds to itself which preserve products and embeddings are classified, along with natural transformations between them. Such functors that are also natural bundles can be thought of as ways of defining infinitesimal neighborhoods for points in all smooth manifolds.

It has become commonplace to think of fibers in any fiber bundle over a manifold M as representing in some sense infinitesimal neighborhoods of points of M. However, in trying to define neighborhoods simultaneously for all manifolds in a natural way, it seems reasonable to assume that they behave well with respect to taking products. This leads to the concept of product-preserving natural bundle (PPNB). Natural bundles on various categories of manifolds have been investigated extensively, for example in [4], [6], and [7]. Here, 'natural bundle' will mean a functor, F, on the category of C^∞ manifolds and C^∞ maps such that for a manifold M, $F(M)$ is a fiber bundle over M; for a map f, $F(f)$ is a bundle map covering f; and such that F is local in the sense that whenever $f=g$ in a neighborhood of a point p, then $F(f)=F(g)$ on the fiber over p. To say that F is product-preserving means that $F(M\times N)$ is naturally equivalent to $F(M)\times F(N)$.

In [2], the k-jet functors $(\cdot)^n_k$ are defined and shown to be PPNB's. For a C^∞ manifold M, $M^n_k=\{k\text{-jets at } 0 \text{ of } C^\infty \text{ maps } \mathbb{R}^n\to M\}$, and for C^∞ maps $f:M\to N$, $g:\mathbb{R}^n\to M$, $f^n_k(j_k(g)_0)=j_k(f\circ g)_0$, where $j_k(\cdot)_0$ indicates the k-jet of a map at 0.

In fact, M^1_1 is just the tangent space of M, with a tangent vector given as a derivative of a curve in M. The fiber of M^1_1 over a point p can be thought of as an infinitesimal neighborhood of p, containing information about first derivatives at p of functions defined on M. Similarly, the fiber of M^n_k over p contains information about partial derivatives at p, in up to n variables, of order k or less.

We will completely classify PPNB's in which the fibers are connected, and will show that each is, in a natural sense, a quotient of some $(\cdot)^n_k$, forgetting information about some of the derivatives.

In fact, we will define and classify 'product-preserving functors', as defined below. We replace the localness condition for natural bundles with the assumption

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that the functor takes an embedding to a one-to-one map. For natural bundles, this is equivalent to localness.

Notations and Terminology. In the following, unless otherwise noted, M and N are C^∞ manifolds (paracompact and non-empty). If $f_i : M_i \to N_i$, $i=1,2$, then $f_1 \times f_2$ is the product map $M_1 \times M_2 \to N_1 \times N_2$. If $M_1 = M_2$, then (f_1, f_2) is the obvious map $M_1 \to N_1 \times N_2$. $C^\infty(M)$ is the ring of C^∞ functions from M to $\mathbb{R}$.

An $\mathbb{R}$-algebra will always mean a commutative, associative ring A with unit, equipped with a unitary ring homomorphism $\mathbb{R} \to A$. If A and B are $\mathbb{R}$-algebras, $\mathrm{Hom}(A,B)$ will denote the $\mathbb{R}$-algebra homomorphisms from A to B.

A functor F (between categories with products) is called product-preserving if whenever $\pi_i : X_1 \times X_2 \to X_i$, $i=1,2$, are the projection maps,

$$(F(\pi_1), F(\pi_2)) : F(X_1 \times X_2) \to F(X_1) \times F(X_2)$$

is an isomorphism. $(F(\pi_1), F(\pi_2))$ will be treated as an identification. Under this identification, $F(\pi_i)$ is itself the projection on the ith factor, and if f_1, f_2 are morphisms, $F(f_1 \times f_2) = F(f_1) \times F(f_2)$. An equivalent definition of 'product preserving' is that $F(X \times Y)$ is naturally equivalent to $F(X) \times F(Y)$.

Definition. A PPF is a functor F from the category of connected C^∞ manifolds and C^∞ maps to itself which is product-preserving in the above sense and such that for any embedding $f : M \to N$, $F(f)$ is one-to-one.

Remarks. It would be sufficient to assume that F takes its values in topological manifolds and continuous maps. We will not use smoothness of $F(M)$ in the classification, although we need the fact that a Euclidean topological group is a Lie group. It will follow from the classification that every PPF nevertheless has a C^∞ structure.

The connectedness of $F(M)$ follows from the connectedness of $F(\mathbb{R})$, which is the only case we will use. In fact, it would even be sufficient to assume only that $F(\mathbb{R})$ has at most countably many components (since a topological $\mathbb{R}$-algebra with countably many components is connected).

The fact that F is defined only on connected manifolds is not essential to the classification. It will be clear that every PPF has an extension to the category of all C^∞ manifolds. If the PPF happens to be a natural bundle, this extension is unique because of the localness property of natural bundles. It is not unique for other PPF's. (See the discussion at the end of the paper.)

Suppose $\{x\}$ is a manifold with a single point. Then so is $F(\{x\})$. For, if $\pi : \{x\} \times \{x\} \to \{x\}$ is the projection, then $F(\pi) : F(\{x\}) \times F(\{x\}) \to F(\{x\})$ is also a projection and is, like π, a homeomorphism. This allows us to define a function $\varphi : \mathbb{R} \to F(\mathbb{R})$.

Definition. If F is a PPF, we define *the cannonical function* $\varphi : \mathbb{R} \to F(\mathbb{R})$ as follows: For $t \in \mathbb{R}$, let $c_t : \{x\} \to \mathbb{R}$ be the map $c_t(x) = t$. Define $\varphi(t)$ to be the image of the map $F(c_t)$.

Proposition 1. *Let F be a PPF. Let $+$ and $*$ be addition and multiplication in $\mathbb{R}$, considered as maps $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Then $F(+)$ and $F(*)$ make $F(\mathbb{R})$ into a ring which is either trivial (the zero ring) or is an $\mathbb{R}$-algebra, with the cannonical map $\varphi : \mathbb{R} \to F(\mathbb{R})$ defining the $\mathbb{R}$-algebra structure.*

Proof. Since the relevant properties can be expressed by diagrams, it is easy to check that $F(\mathbb{R})$ is a commutative, associative ring with additive identity $\varphi(0)$ and multiplicative identity $\varphi(1)$. Also, φ is a homomorphism since $\varphi(s+t) = F(c_{t+s})(\cdot) = F(+ \circ (c_t, c_s))(\cdot) = F(+)(\varphi(t), \varphi(s)) = \varphi(t) + \varphi(s)$, and similarly for multiplication.

If $\varphi(0) = \varphi(1)$, then the additive and multiplicative identities of $F(\mathbb{R})$ are equal, and $F(\mathbb{R})$ is trivial. Otherwise, φ must be injective and $F(\mathbb{R})$ is an $\mathbb{R}$-algebra. $\square$

Suppose $F(\mathbb{R})$ is trivial. If M is any manifold and $f : M \to \mathbb{R}^n$ is an embedding, then the fact that $F(f) : F(M) \to F(\mathbb{R}) \times \cdots \times F(\mathbb{R})$ is one-to-one shows that $F(M)$ also consists of a single point. We say that F is trivial. From now on, F will be some fixed non-trivial PPF.

Proposition 2. *The canonical map $\varphi : \mathbb{R} \to F(\mathbb{R})$ is continuous.*

Proof. Consider $F(\mathbb{R})$ as a Lie group under addition, and let $\exp : \mathbb{R}^n \to F(\mathbb{R})$ be the exponential map, where we have identified the Lie algebra of $F(\mathbb{R})$ with $\mathbb{R}^n$. Since $F(\mathbb{R})$ is connected and abelian, there is an X such that $\exp(X) = \varphi(1)$. Let $\psi : \mathbb{R} \to F(\mathbb{R})$ be the continuous map $\psi(t) = \exp(tX)$. Since ψ is additive, it follows that $\psi(q) = \varphi(q)$ for $q \in \mathbb{Q}$. I claim that in fact, $\varphi = \psi$.

Let $r \in \mathbb{R}$. Choose $q_i \in \mathbb{Q}$, $i = 1, 2, \ldots$ and a C^∞ function $f : \mathbb{R} \to \mathbb{R}$ such that $f(1/i) = q_i$ and $f(0) = r$. (One way to ensure that this can be done is to choose q_i so that $|q_i - r| < e^{-i^2}$. Then all derivatives of f at 0 will be 0.)

Now, $F(f)$ and ψ are continous, so $\lim F(f)(\psi(1/i)) = F(f)(\psi(0))$. Since $\psi(q) = \varphi(q)$ for $q \in \mathbb{R}$, this is the same as $\lim F(f)(\varphi(1/i)) = F(f)(\varphi(0))$. Now, $\varphi(t) = F(c_t)(\cdot)$, as in the definition of φ, and it follows that $F(f)(\varphi(t)) = \varphi(f(0))$. Thus, $\lim \varphi(f(1/i)) = \varphi(f(0))$, or $\lim \varphi(q_i) = \varphi(r)$. Finally, since $\varphi(q_i) = \psi(q_i)$, we get $\psi(r) = \lim \psi(q_i) = \varphi(r)$. $\square$

Definition. Let $a \in F(M)$. We define an $\mathbb{R}$-algebra homomorphism $h_a : C^\infty(M) \to F(\mathbb{R})$ by $h_a(f) = F(f)(a)$. It is easy to check that h_a is in fact a homomorphism.

If $a \in F(M)$ and $f : M \to N$, a computation shows that $h_{F(f)(a)} = h_a \circ f^*$, where $f^* : C^\infty(N) \to C^\infty(M)$ is the map $f^*(g) = g \circ f$.

Proposition 3. *$F(\mathbb{R})$ is a finite-dimensional $\mathbb{R}$-algebra with its usual vector space topology. Furthermore, all quotient fields of $F(\mathbb{R})$ are equal to $\mathbb{R}$. Thus, $F(\mathbb{R})$ is a product of local, finite-dimensional $\mathbb{R}$-algebras with quotient field $\mathbb{R}$.*

Proof. Proposition 2 implies that the action of $\mathbb{R}$ on $F(\mathbb{R})$ is continuous. So, the exponential map for $F(\mathbb{R})$, which is additive and so preserves the action of $\mathbb{Q}$, also preserves the action of $\mathbb{R}$, and thus is a vector space isomorphism. This proves the first statement.

Any quotient field of $F(\mathbb{R})$ must be $\mathbb{R}$ or $\mathbb{C}$. Suppose $h: F(\mathbb{R}) \to \mathbb{C}$ is a surjective homomorphism. Let $a \in F(\mathbb{R})$ be such that $h(a) = i$. Then $h \circ h_a : C^\infty(\mathbb{R}) \to \mathbb{C}$ is also surjective, since $h_a(\mathrm{id}_{\mathbb{R}}) = a$. But this is impossible since then $h \circ h^a(\mathrm{id}_{\mathbb{R}}^2 + 1) = 0$, and $\mathrm{id}_{\mathbb{R}}^2 + 1$ is a unit in $C^\infty(\mathbb{R})$. For the last statement, see [1, p. 90]. □

Notation. Write $F(\mathbb{R}) = A_1 \times \cdots \times A_m$, where each A_i is local. Let $\mathfrak{m}_i$ be the maximal ideal of A_i. Let $\sigma_i : A_1 \times \cdots \times A_m \to \mathbb{R}$ be the map $\sigma_i(a_1, \ldots, a_m) = a_i \bmod \mathfrak{m}_i$.

Lemma. *If $\mathfrak{m}$ is a maximal ideal of $C^\infty(M)$ with $C^\infty(M)/\mathfrak{m} = \mathbb{R}$, then there is an $x \in M$ such that $\mathfrak{m} = \mathfrak{m}_x = \{f \in C^\infty(M) \mid f(x) = 0\}$.*

Proof (adapted from [5]). Construct $g: M \to \mathbb{R}$ so that $g^{-1}(t)$ is compact for each $t \in \mathbb{R}$. Let $h: C^\infty(M) \to \mathbb{R}$ be the quotient map and let $r = h(g)$. Then $h(g - r) = 0$ and $(g - r)^{-1}(0)$ is compact. It follows by a standard argument that $\mathfrak{m} = \mathfrak{m}_x$ for some $x \in (g - r)^{-1}(0)$. □

Proposition 4. *Let $h \in \mathrm{Hom}(C^\infty(M), F(\mathbb{R}))$. Then for each $i = 1, \ldots, m$, the kernal of $\sigma_i \circ h$ is of the form $\mathfrak{m}_{x_i} = \{f \in C^\infty(M) \mid f(x_i) = 0\}$ for some $x_i \in M$. Furthermore, there is an integer k, depending only on $F(\mathbb{R})$, such that for $f \in C^\infty(M)$, $h(f)$ depends only on the k-jets of f at $x_1, \ldots, x_m$.*

Proof. The first statement follows from the lemma. For the second, choose k so that $\mathfrak{m}_i^k = 0$ for $i = 1, \ldots, m$. (For the existence of such a k, see [1, p. 90].) □

Proposition 5. *The map $a \mapsto h_a$ from $F(M)$ to $\mathrm{Hom}(C^\infty(M), F(\mathbb{R}))$ is a bijection.*

Proof. First, suppose $M = \mathbb{R}^n$. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the ith projection. For $a \in F(\mathbb{R}^n)$, we can write $a = (a_1, \ldots, a_n)$, where $a_i \in F(\mathbb{R})$. Let $a, b \in F(\mathbb{R}^n)$. If $h_a = h_b$, then $h_a(\pi_i) = h_b(\pi_i)$, $i = 1, \ldots, n$, so $a = b$. Now, suppose $h \in \mathrm{Hom}(C^\infty(\mathbb{R}^n), F(\mathbb{R}))$. Let $a_i = h(\pi_i)$ and $a = (a_1, \ldots, a_n)$. Then $h = h_a$: In fact $h(\pi_i) = h_a(\pi_i) = a_i$ and so h and h_a agree on any polynomial function. Since a polynomial can be found with any specified k-jets at finitely many points, the claim follows from Proposition 4. This proves the proposition in the case $M = \mathbb{R}^n$.

Now, let M be any connected manifold. There is an embedding $i: M \to \mathbb{R}^n$ for some n. Using the first part of the proof and the fact that i is, by the definition of

PPF, one-to-one, we verify that $a \mapsto h_a$ is one-to-one. (Note that even if we were not assuming M to be connected, this map would still be one-to-one. The same proof works if M has only finitely many components, but for any given homomorphism $C^\infty(M) \to F(\mathbb{R})$, only finitely many components are involved.)

Finally, to show the map is onto, let $h \in \mathrm{Hom}(C^\infty(M), F(\mathbb{R}))$. Let x_i, $i = 1, \ldots, m$, be as in the proof of Proposition 4. Let $f: \mathbb{R}^N \to M$, where $N = \dim(M)$, be an open embedding whose image contains $x_1, \ldots, x_m$. Choose $g: M \to \mathbb{R}^N$ such that $f \circ g$ is the identity on an open set containing all the x_i. Then $h \circ (f \circ g)^* = h$. By the first part of the proof, $h \circ g^* = h_a$ for some $a \in F(\mathbb{R}^N)$. But then, $h = h \circ g^* \circ f^* = h_a \circ f^* = h_{F(f)(a)}$. □

Recall that $F(\mathbb{R}^n) = F(\mathbb{R}) \times \cdots \times F(\mathbb{R})$ is a vector space with its usual topology. The above proof shows that if $f: \mathbb{R}^n \to M$ is a coordinate chart on M, then $F(f)$ is a coordinate chart on $F(M)$. Since $F(M)$ is covered by such charts, they determine its C^∞ structure.

Furthermore, if A is any finite-dimensional $\mathbb{R}$-algebra, all of whose quotient fields are $\mathbb{R}$, we can define a PPF F_A by

$$F_A(M) = \mathrm{Hom}(C^\infty(M), A), \qquad F(f)(h) = h \circ f^*.$$

The C^∞ structure on $F_A(M)$ can be defined using charts of the kind discussed above. Note that $F_A(\mathbb{R})$ is isomorphic to A. In fact, Proposition 5 allows us to identify F with $F_{F(\mathbb{R})}$.

The correspondences $F \mapsto F(\mathbb{R})$ and $A \mapsto F_A$ between PPF's and $\mathbb{R}$-algebras with the given properties actually give a natural equivalence of categories, once we discuss the appropriate morphisms between PPF's.

If F and G are PPF's, a natural transformation T from F to G is a collection of maps $T(M): F(M) \to G(M)$ such that all diagrams

$$\begin{tikzcd} F(M) \arrow[r, "T(M)"] \arrow[d] & G(M) \arrow[d] \\ F(N) \arrow[r, "T(N)"] & G(N) \end{tikzcd}$$

commute. (It turns out that the maps $T(M)$ are automatically smooth.) It is easy to check that $T(M \times N) = T(M) \times T(N)$ and, using this, that $T(\mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism from $F(\mathbb{R})$ to $G(\mathbb{R})$.

Furthermore, if $s: F(\mathbb{R}) \to G(\mathbb{R})$ is any $\mathbb{R}$-algebra homomorphism, s extends to a natural transformation S from F to G by $S(M)(h) = s \circ h$, for $h \in F(M)$, which we have here identified with $\mathrm{Hom}(C^\infty(M), F(\mathbb{R}))$. This extension is unique: The definition for $\mathbb{R}$ forces that for $\mathbb{R}^n$, and hence for any embedded subset of $\mathbb{R}^n$ – that is, for all M. $S(\mathbb{R}^n)$ is C^∞ since it is linear. The fact that $S(M)$ is C^∞ can be checked using the charts on $F(M)$ and $G(M)$ defined above.

Now, suppose $F(\mathbb{R})$ is local, with maximal ideal $\mathfrak{m}$. Then F is a natural bundle: the projection $\pi: F(M) \to M$ can be described as the natural transformation from F to the identity PPF corresponding to the homomorphism $\sigma: F(\mathbb{R}) \to \mathbb{R}$. Alternatively, $\pi(h) = x$, where $\mathfrak{m}_x = \ker(\sigma \circ h)$. If $f, g: M \to N$ and $f = g$ on a neighborhood of x, then $F(f) = F(g)$ on $\pi^{-1}(x)$ by Proposition 4, so F is local. Thus, F is a PPNB.

In the general case, $F(\mathbb{R})$ is a product $A_1 \times \cdots \times A_m$ of local algebras. There are PPNB's F_{A_i} with $F_{A_i}(\mathbb{R}) = A_i$. It is clear from the above that F is a product $F_{A_i} \times \cdots \times F_{A_m}$ in the obvious sense. It is also clear that if $m > 1$, then F is not a PPNB.

We can sum up all our results in the following classification theorem.

Theorem. *There is a one-to-one correspondence, up to equivalence, between PPF's and finite-dimensional $\mathbb{R}$-algebras all of whose quotient fields are $\mathbb{R}$, given by $F \mapsto F(\mathbb{R})$. If F and G are PPF's, then there is a one-to-one correspondence between natural transformations from F to G and* $\mathrm{Hom}(F(\mathbb{R}), G(\mathbb{R}))$, *given by $T \mapsto T(\mathbb{R})$. In this equivalence, PPNB's correspond exactly to local algebras. Every PPF is a product of PPNB's.* □

Applications and discussion

(1) The algebra corresponding to the functor $(\cdot)^n_k$ is $\mathbb{R}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{k+1}$. Since any local, finite-dimensional $\mathbb{R}$-algebra with quotient field $\mathbb{R}$ is a quotient of such an algebra, we see that every PPNB is a quotient of $(\cdot)^n_k$ for some n, k. Thus, every PPNB is a 'jet functor'.

(2) Let F and G be PPF's. Then $F \circ G$ is also a PPF. Some calculations involving a choice of basis show that $F(G(\mathbb{R}))$ is isomorphic as an $\mathbb{R}$-algebra to $F(\mathbb{R}) \otimes G(\mathbb{R})$. Thus, in particular, by the classification theorem, $F \circ G$ is naturally equivalent to $G \circ F$. For example, if T is the tangent bundle, then $F(T(M)) = T(F(M))$. This in turn implies that if H is a Lie group with Lie algebra $\mathfrak{h}$ and exponential map exp, then $F(H)$ is a Lie group with Lie algebra $F(\mathfrak{h})$ and exponential map $F(\exp)$. Furthermore, F takes representations of H to representations of $F(H)$. In [3], I study such representations in the case $F = (\cdot)^n_k$, which arises, as explained in [2], in the study of gauge theories.

(3) Let F be a PPF with $F(\mathbb{R}^n) = A_1 \times \cdots \times A_m$, where A_i is local. One can associate with A_i an 'infinitesimal manifold' D_i with $C^\infty(D_i) = A_i$, and identify $\mathrm{Hom}(C^\infty(M), A_i)$ with 'maps' from D_i to M. Letting D be the disjoint union of $D_1, \ldots, D_m$, we can then identify $F(M)$ with M^D, maps from D to M. The classification theorem then says that all PPF's are of this form, and furthermore that if D and E are two such infinitesimal manifolds, then natural transformations from the PPF $(\cdot)^D$ to the PPF $(\cdot)^E$ correspond to maps from E to D.

(4) Finally, we return to a discussion of PPF's on arbitrary, rather than just connected, manifolds. We can give a complete classification in this case as well. If F

is any PPF (on connected manifolds), we can extend F to arbitrary manifolds in the obvious way: $F(M)=\mathrm{Hom}(C^\infty(M), F(\mathbb{R}))$. However, this extension is not unique, unless F is a PPNB. For example, consider the PPF $P_m(M)=M\times\cdots\times M$ (m times). If for an arbitrary manifold M, we define $P_m^c(M)$ to be the disjoint union of $\{P_m(N)\,|\,N \text{ is a connected component of } M\}$, then P_m^c is a PPF which is an alternate extension of P_m to arbitrary manifolds. Returning to the PPF F, extended canonically to arbitrary manifolds, we have the projection $\pi: F(M)\to P_m(M)$, where m is the number of local factors of $F(\mathbb{R})$. Let $F^c(M)=\pi^{-1}(P_m^c(M))$. Then F^c is another PPF which extends the original F.

I claim that any PPF on arbitrary manifolds is a product $F_1^c\times\cdots\times F_k^c$ for some PPF's $F_1,\ldots,F_k$ on connected manifolds. Let G be a PPF on arbitrary manifolds. Suppose $G(\mathbb{R})=A_1\times\cdots\times A_m$, where A_i is local. Define $F(M)=\mathrm{Hom}(C^\infty(M), G(\mathbb{R}))$, and let $\pi: F(M)\to P_m(M)$ be the projection, as above. As remarked in the proof of Proposition 5, that proof shows that the map $G(M)\to F(M)$ taking a to h_a is an injection. So the claim that G is of the stated form can be translated into the claim that the set $\{1,2,\ldots,m\}$ can be divided into equivalence classes $I_1,\ldots,I_k$ such that $G(M)=\pi^{-1}(\bigcup\{N_1\times\cdots\times N_m\,|\,N_1,\ldots,N_m$ are connected components of M and for $r=1,\ldots,k$, $s,t\in I_r$ implies $N_s=N_t\})$. (Each equivalence class I_r gives an F_r, namely the PPF corresponding to the product of the A_s for $s\in I_r$.)

To discover the equivalence classes, consider $G(X)$, where $X=\{1,\ldots,m\}$, considered as a discrete manifold. Now $G(X)\subseteq F(X)=X^m$. Choose $(i_1,\ldots,i_m)\in G(X)$ with the maximum number of distinct members. The equivalence classes I_r will then be the non-empty sets of the form $\{s\,|\,i_s=n\}$, for $n=1,2,\ldots,m$. We assume there are k such classes.

Now, let D be a discrete manifold. In this case, our claim becomes that $G(D)=\{(d_1,\ldots,d_m)\in D^m\,|\,\text{for } r=1,\ldots,m,\ s,t\in I_r \text{ implies } d_s=d_t\}$. Suppose not. That is, there exist r,s,t such that $s,t\in I_r$ and $d_s\neq d_t$. This means that among the pairs $(i_1,d_1),\ldots,(i_m,d_m)$, there are at least $k+1$ distinct pairs. If $f: X\times D\to X$ is any function that maps these $k+1$ pairs onto the integers $1,2,\ldots,k+1$, then $G(f)((i_1,\ldots,i_m),(d_1,\ldots,d_m))$, which is just equal to $(f(i_1,d_1),\ldots,f(i_m,d_m))$, has at least $k+1$ distinct members, contradicting the maximality of k. This proves the claim for D. And it follows immediately that $G(\mathbb{R}^n\times D)=G(\mathbb{R}^n)\times G(D)=F(\mathbb{R}^n)\times G(D)$ is also of the proper form, since the components of $\mathbb{R}^n\times D$ just correspond to the points of D.

Turning to the general case, let M be any manifold. Let D be a discrete manifold with one point for each component of M, and let $f: M\to D$ be the obvious map. If $a\in G(M)$, $G(f)(a)$ just classifies the component of P_m over which a lies, and it follows that this component must be of the right form. Next, suppose that $x_1,\ldots,x_m$ are points of M such that for $r=1,\ldots,k$, $s,t\in I_r$ implies x_s,x_t are in the same component, say C_r, of M. We will be done if we show that $G(M)$ includes the fiber $\pi^{-1}(x_1,\ldots,x_m)\subseteq F(M)$. Let $N=\mathbb{R}^n\times\{1,\ldots,k\}$, where n is the maximum of the dimensions of $C_1,\ldots,C_k$. We can choose $y_1,\ldots,y_m\in N$ and a map $g: N\to M$

such that $g(y_i)=x_i$ and g is a submersion on neighborhoods of the y_i. We then know from the exact description of F that $F(g)(\pi^{-1}(y_1,\dots,y_m))=\pi^{-1}(x_1,\dots,x_m)$. We also know from the above that $G(N)$ includes $\pi^{-1}(y_1,\dots,y_m)$. So we are done.

With regard to remark (3) above, we can still consider $G(M)$ to consist of the maps on some sort of infinitesimal maniold, in the following sense. As before, let D_i be the 'manifold' with coordinate ring A_i. The D_i are to be divided into classes $\{D_i \mid i\in I_r\}$, $r=1,\dots,k$. D is just the union of all D_i, with the provision that any map from D to a manifold M must take all the D_i in any equivalence class to the same component of M. Then $G(M)=M^D$. What makes this more interesting is that we can still say that natural transformations between PPF's on arbitrary manifolds correspond to maps between the associated infinitesimal manifolds, where these maps are required to respect the equivalence classes.

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